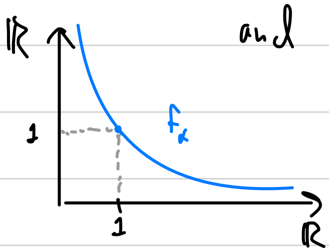
Math 564: Advance Analysis 1
Lecture 26

For $p<q$, what is the relationship between L^p and L^q?

Example. Let $f_\alpha : (0,\infty) \to (0,\infty)$ be the function $x \mapsto x^{-\alpha}$, for a fixed $\alpha > 0$. Note that, by our knowledge of Riemann integral, $\mathbb{1}_{(0,1)} \cdot f \in L^p \iff p < \frac{1}{\alpha}$ and $\mathbb{1}_{[1,\infty)} \cdot f \in L^p \iff p > \frac{1}{\alpha}$. So for $p<q$, the functions in L^p are blowing up faster at a point, while functions in L^q are decaying slower at ∞.



The first case only matters when there are arbitrarily small measure sets, i.e. the measure is atomless, while second phenomenon only happens in infinite measure spaces. We can make this precise:

Prop. Let A be a set and $0 < p < q \le \infty$. Then $\ell^p(A) \subseteq \ell^q(A)$, in fact, $\|f\|_q \le \|f\|_p$, for any $f \in \ell^p(A)$.

Proof. $q = \infty$. $\|f\|_\infty = |f(a)|$ for some $a \in A$, so $\|f\|_\infty \le \left(\sum_{a\in A} |f(a)|^p\right)^{\frac{1}{p}} = \|f\|_p < \infty$.

$q<\infty$.
$$\|f\|_q = \left(\sum_{a\in A} |f(a)|^q\right)^{\frac{1}{q}} = \left(\sum_{a\in A} |f(a)|^p \cdot |f(a)|^{q-p}\right)^{\frac{1}{q}} \le \|f\|_\infty^{\frac{q-p}{q}} \cdot \left(\sum_{a\in A} |f(a)|^p\right)^{\frac{1}{q}}$$
$$\le \|f\|_p^{1-\frac{p}{q}} \cdot \|f\|_p^{\frac{p}{q}} = \|f\|_p. \qquad \square$$

Prop. Let (X,μ) be a finite measure space and $0 < p < q \le \infty$. Then $L^q(X,\mu) \subseteq L^p(X,\mu)$ and $\|f\|_p \le \|f\|_q \cdot \mu(X)^{\frac{1}{p}-\frac{1}{q}}$.

Proof. $q=\infty$.
$$\|f\|_p = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}} \le \left(\|f\|_\infty^p \cdot \mu(X)\right)^{\frac{1}{p}} = \|f\|_\infty \cdot \mu(X)^{\frac{1}{p}-\frac{1}{\infty}}.$$

$q<\infty$.
$$\|f\|_p = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}} = \left(\int |f|^p \cdot 1 \, d\mu\right)^{\frac{1}{p}} \le \left(\||f|^p\|_{q/p} \cdot \|1\|_{q/(q-p)}\right)^{\frac{1}{p}} =$$

Hölder with $\frac{q}{p}$, $\left(1-\frac{p}{q}\right)^{-1} = \frac{q}{q-p}$

$$\left(\left(\int |f|^{p\cdot\frac{q}{p}}\,d\mu\right)^{\frac{p}{q}}\cdot \mu(X)^{\frac{q-p}{q}}\right)^{\frac{1}{p}} = \left(\|f\|_q^p \cdot \mu(X)^{1-\frac{p}{q}}\right)^{\frac{1}{p}} = \|f\|_q \cdot \mu(X)^{\frac{1}{p}-\frac{1}{q}}. \quad \square$$

Duality in L^p spaces. For a normed vector space $(X, \|\cdot\|)$, the dual of X, denoted by X^*, is the space of bdd linear functionals

$$\Lambda : X \to \mathbb{R}$$

with the operator norm, i.e. $\|\Lambda\| := \sup_{\|f\|=1} |\Lambda f|$.

For inner product spaces, i.e. $(X, \langle\cdot,\cdot\rangle)$, we know that for $y \in X$, the map $x \mapsto \langle x, y\rangle$ is a "typical" linear functional, and X is finite dimensional, these are all linear functionals.
Recall that, say on $C([0,1])$, we can define inner product by
$$\langle f, g\rangle := \int f\bar{g}\,d\lambda.$$

We can do a similar construction with L^p spaces, but instead of taking both f and g from the same space, we take them from conjugate exponent spaces $f \in L^p$ & $g \in L^q$, so that $f\cdot\bar{g}$ is integrable, by Hölder.

Def. For $p \geq 1$, let q be its conjugate exponent and let $g \in L^q$. Then $\Lambda_g : L^p \to \mathbb{R}$ defined by
$$\Lambda_g(f) := \int f\bar{g}\,d\mu$$
is a bdd linear functional with norm $\|\Lambda_g\| \leq \|g\|_q$ hence $|\Lambda_g(f)| \leq \|f\|_p \cdot \|g\|_q$.

Prop. In fact, $\|\Lambda_g\| = \|g\|_q$, so $g \mapsto \Lambda_g : L^q \to (L^p)^*$ is a linear isometry.

Proof. We cook up a function f with $\|f\|_p = 1$ so that $|\Lambda_g(f)| = \|g\|_q$.

Let $f := \dfrac{|g|^{\frac{q}{p}}\,\overline{\mathrm{sgn}(g)}}{\|g\|_q^{q/p}}$, then $\|f\|_p = \left(\dfrac{1}{\|g\|_q^q}\int |g|^q\right)^{\frac{1}{p}} = \left(\dfrac{\|g\|^q}{\|g\|^q}\right)^{\frac{1}{p}} = 1.$

Then

$$|\Lambda_g(f)| = \left|\int f\bar{g}\,d\mu\right| = \int \frac{|g|^{q/p}\cdot|g|}{\|g\|_q^{q/p}}\,d\mu = \frac{1}{\|g\|_q^{q/p}}\cdot\int |g|^q\,d\mu = \frac{1}{\|g\|_q^{q/p}}\cdot\|g\|_q^q$$

$$= \frac{1}{\|g\|_q^{q-1}}\cdot\|g\|_q^q = \frac{1}{\|g\|_q^{-1}} = \|g\|_q.$$

$\square$

Thm. For $1 \le p < \infty$, $(L^p)^* \cong L^q$, for σ-finite spaces (X,μ), where q is the conjugate exp. of p. In fact, the map $g \mapsto \Lambda_g : L^q \to (L^p)^*$ is a linear norm isomorphism.

Proof. Given an arbitrary $\Lambda \in (L^p)^*$, we need to come up with a function $g \in L^q$ so that $\Lambda = \Lambda_g$. Suppose for now that μ is finite. We use Λ to define a signed measure ν on X by

$$\nu(A) := \Lambda(\mathbb{1}_A)$$

for all $A \subseteq X$. By linearity, this is finitely additive, and it is ctbly additive by the continuity of Λ using the fact that $\mu < \infty$ (hence if (A_n) is a sequence of disjoint sets, $\sum_n \mu(A_n) \le \mu(X) < \infty$). Then $\nu = \nu_+ - \nu_-$ for actual measures ν_+, ν_- on X, which are also finite. For a set A, if A is μ-null, then $\mathbb{1}_A = 0$ inside L^p, so $\Lambda(\mathbb{1}_A) = 0$, hence it follows that $\nu_+(A) = \nu_-(A) = 0$ (by splitting $X = X_+ \sqcup X_-$, as usual). In other words, $\nu_+, \nu_- \ll \mu$, so there are Radon-Nikodym derivatives $g_+, g_- : X \to \mathbb{R}$, hence taking $g := g_+ - g_-$, we have $\forall A \subseteq X$,

$$\Lambda(\mathbb{1}_A) = \nu(A) = \int \mathbb{1}_A g\,d\mu.$$

Thus by an approximation argument, $\underset{\Lambda(f)}{\underset{\|}{\int f\,d\nu}} = \int fg\,d\mu$ for

all $f \in L^p$, using DCT and the linearity & continuity of Λ.
One also shows that $g \in L^q$, which follows from the fact that if $\mathbb{1}_A \cdot g \in L^1$ for all $A \subseteq X$, then $g \in L^q$.
Thus, $\Lambda_g = \Lambda$. $\square$

What about L^∞, what is $(L^\infty)^*$? This is a very large space, the space of signed means on (X, μ), which contains all finitely additive probability measures on the measurable space $(X, \mathcal{B})$, including all ultrafilters on $(X, \mathcal{B})$.
In particular, $(\ell^\infty)^*$ contains all ultrafilters on $\mathbb{N}$, whose existence doesn't follow from ZF, we need Choice.